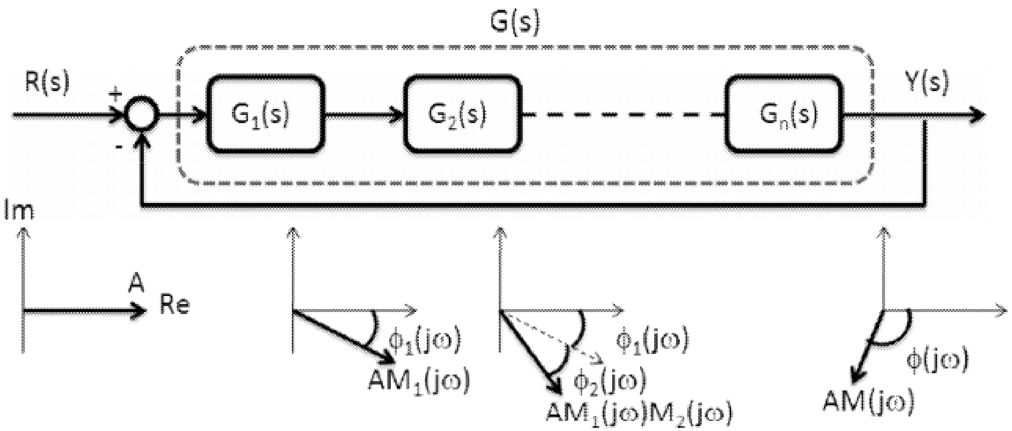
8. Frequency Domain Design

EN 2142 Electronic Control Systems

Dr. Rohan Munasinghe
BSc, MSc, PhD, MIEEE
Department of Electronic and Telecommunication Engineering
Faculty of Engineering
University of Moratuwa 10400

Gain and Phase Contribution



Gain $M_1(j\omega), M_2(j\omega), \ldots M_n(j\omega)$

Phase $\phi_1(j\omega), \phi_2(j\omega), \ldots \phi_n(j\omega)$

Open Loop Analysis

Open Loop Gain

$$\begin{aligned} M(j\omega) &= \Pi_1^n M_i(j\omega) \\ |G(j\omega)|dB &= \Sigma_1^n |G_i(j\omega)|\text{dB} \end{aligned}$$

Open Loop Phase

$$\begin{aligned} \phi(j\omega) &= \Sigma_1^n \phi_i(j\omega) \\ \angle G(j\omega) &= \Sigma_1^n \angle G_i(j\omega) \end{aligned}$$

Phase and gain of common blocks

- **Pole/zero at origin**
- **First order block (zero/pole)**
- **Second order block (pair of poles/zeros)**

Pole/Zero at Origin

- Block Transfer Function $s^{\pm n}$
- Gain of the block

$$\begin{aligned} |G_i(j\omega)| &= \omega^{\pm n} \\ |G_i(j\omega)|dB &= \pm n 20 \log \omega \text{dB} \end{aligned}$$

- Phase of the block

$$\begin{aligned} \angle G_i(j\omega) &= \pm n \tan^{-1}\left(\frac{\omega}{0}\right) \\ &= \pm 90 n^0 \end{aligned}$$

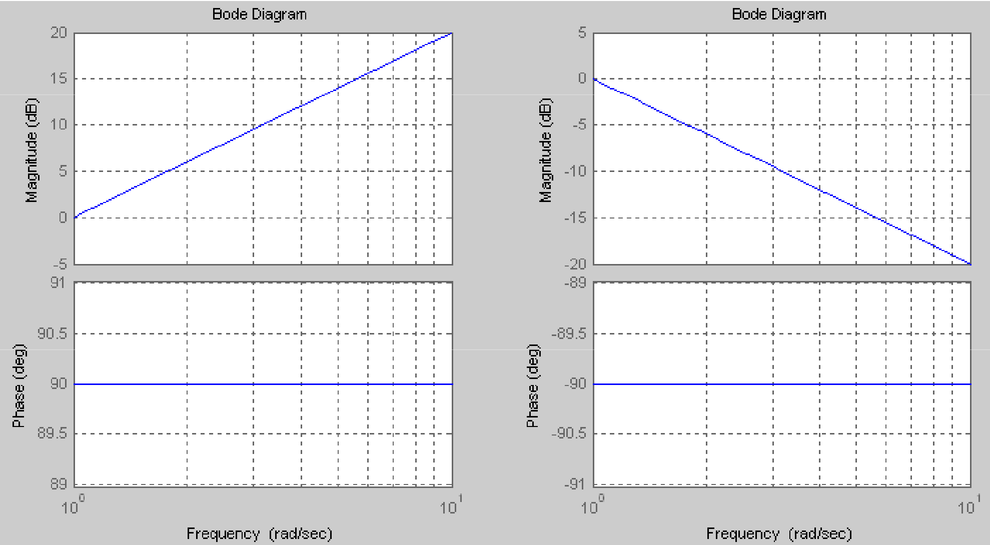
```
% Bode plots of a zero and a pole

% for a zero at origin
subplot(121); sys=tf([1 0],[0 1]); bode(sys); grid on;

% for a pole at origin
subplot(122); sys=tf([0 1],[1 0]); bode(sys); grid on;
```

Pole/Zero at Origin

Zero at origin　　　　Pole at origin



First Order Block (zero/pole)

- Pole or a zero (single or multiple) $\frac{s+z}{1}$ $\frac{1}{s+p}$
- Gain $|G(j\omega)| = |(j\omega + a)|^{\pm 1}$ +1 for zero, -1 for pole

$$|G(j\omega)|\text{dB} = \pm 20\log(\sqrt{\omega^2 + a^2})\text{dB}$$

- Phase $\angle G(j\omega) = \pm\tan^{-1}\left(\frac{\omega}{a}\right)$

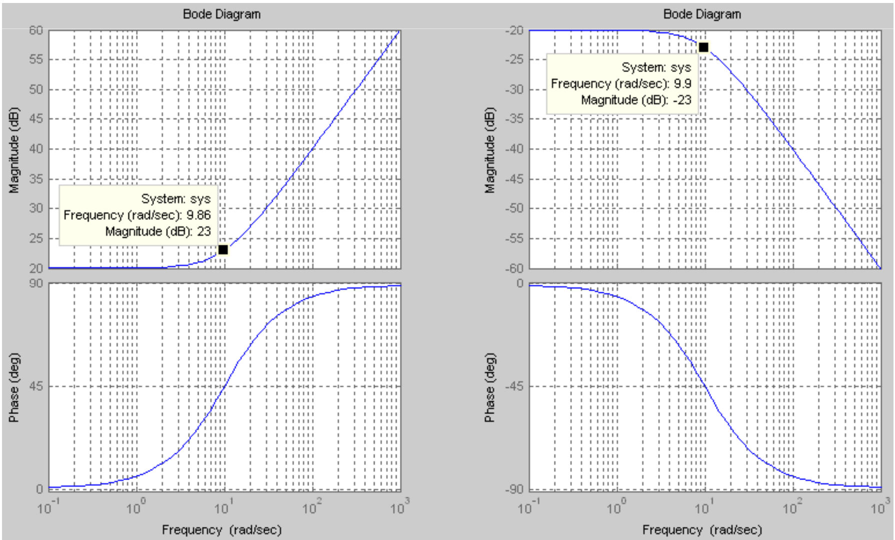
Gain at milestone frequencies $\frac{1}{s+a}$ or $\frac{s+a}{1}$

$$\begin{aligned}|G(j0)| &= \pm 20\log\sqrt{0^2 + a^2} \\ &= \pm 20\log a\text{dB} \\ |G(ja)| &= \pm 20\log\sqrt{a^2 + a^2} = \pm 20\log\sqrt{(2a^2)}\text{dB} \\ &= \pm(20\log a + 20\log\sqrt{2})\text{dB} \\ &= \pm 20\log a\text{dB} \pm 3\text{dB}\end{aligned}$$

First Order Block (zero and pole)

```
% for a zero at origin
subplot(121); sys=tf([1 10],[0 1]); bode(sys); grid on;

% for a pole at origin
subplot(122); sys=tf([0 1],[1 10]); bode(sys); grid on;
```



Second Order Block

Pair of zeros $\frac{s^2 + 2\varsigma\omega_n + \omega_n^2}{1}$　　Pair of poles $\frac{1}{s^2 + 2\varsigma\omega_n + \omega_n^2}$

- Gain

$$\begin{aligned}|G(j\omega)| &= |s^2 + 2\zeta\omega_n s + \omega_n^2|^{\pm 1} \\ &= |(\omega_n^2 - \omega^2) + j2\zeta\omega\omega_n|^{\pm 1} \\ &= \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \\ |G(j\omega)|\text{dB} &= \pm 20\log\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\text{dB}\end{aligned}$$

- Phase $\angle G(j\omega) = \pm\tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$

Second Order Block

- Gain at milestone frequencies

$$|G(j0)|\text{dB} = \pm 20\log\sqrt{\omega_n^4 + 0^2}\text{dB}$$
$$= \pm 40\log\omega_n\text{dB}$$
$$|G(j\omega_n)|\text{dB} = \pm 20\log\sqrt{0^2 + (2\zeta\omega_n^2)^2}\text{dB}$$
$$= \pm 20\log 2\zeta\omega_n^2\text{dB}$$
$$= \pm(40\log\omega_n + 20\log 2\zeta)\text{dB}$$
$$= \pm 40\log\omega_n\text{dB} \pm \underbrace{20\log 2\zeta\text{dB}}$$

Gain change as frequency changes 0 → ω_n

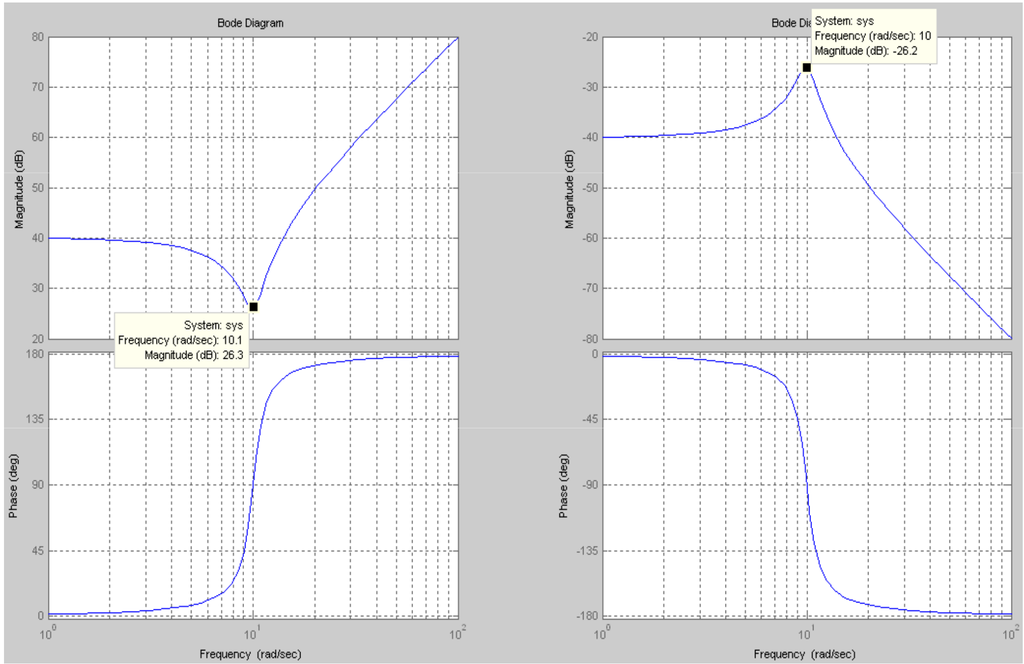
```
zeta=0.1;
omegan=10;

% for a pair of
subplot(121);
sys=tf([1 2*zeta*omegan omegan^2],[1]);
bode(sys);
grid on; hold on;

% for a pair of poles
subplot(122);
sys=tf([1],[1 2*zeta*omegan omegan^2]);
bode(sys);
grid on; hold on;
```

$$|G(j\omega)|\text{dB} = \pm 20\log\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}\text{dB}$$

Second Order Block



Phase Margin and Gain Margin

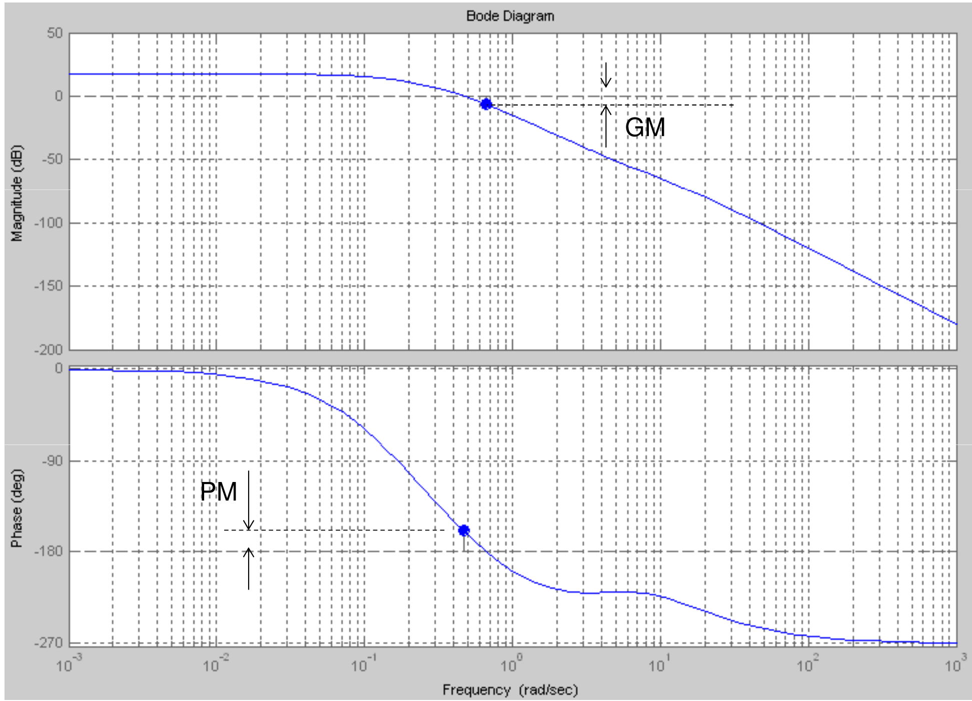
- Consider the open loop transfer function

$$\frac{(s+5)(s+6)}{(s^3+14s^2+13s+2)(s^2+10s+2)}$$

```
% System transfer function
num=conv([1 5],[1 6]);
den=conv([1 14 13 2],[1 10 2]);
sys=tf(num,den);

% Bode plots
% calcuates gain and phase at 5rad/s
[gain,phase]=bode(sys,5)
bode(sys); grid on;
```

Gain Margin and Phase Margin



Servo Controller Design

Servo controller design needs to adjust the following

- Open Loop Bandwidth (BW)
- Phase margin at BW frequency
- Steady state error

Servo Controller Design

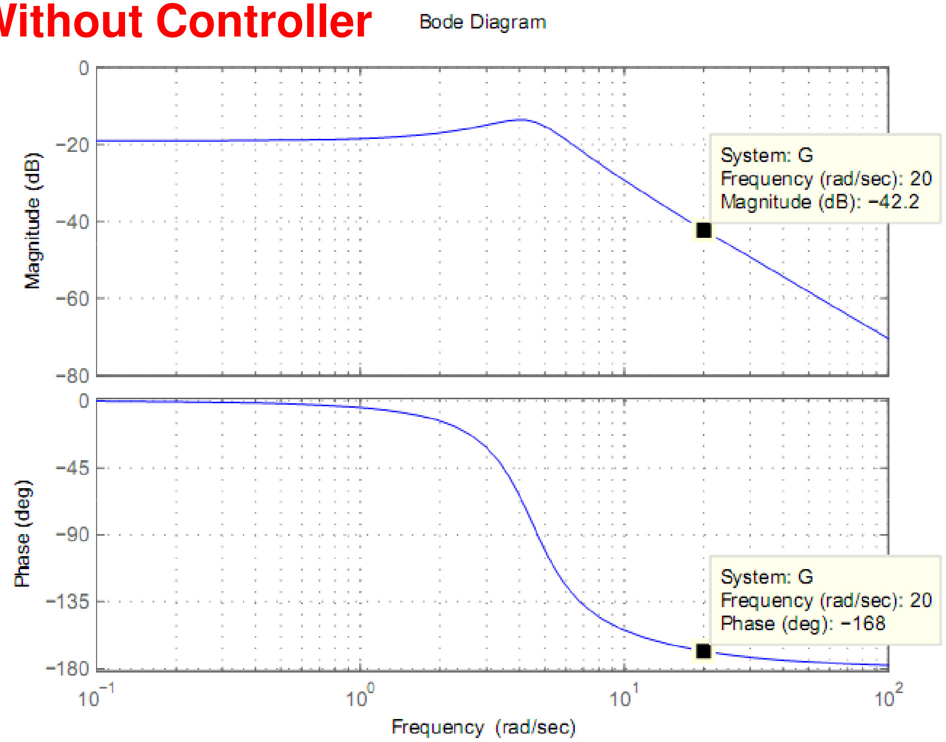
Example $G(s) = \frac{3(s+3)}{(s+4)(s^2+3s+20)}$

Design C(s) to have BW=20[rad/s], PM=45°, and Unit step steady state error=0.01

Step 1: Draw gain and phase plots

```
numG=3*[1 3];
denG=conv([1 4],[1 3 20]);
G=tf(numG,denG);
bode(G); grid on;
[gainG,phaseG]=bode(G,20) % gain and phase at 20[rad/s]
```

Without Controller

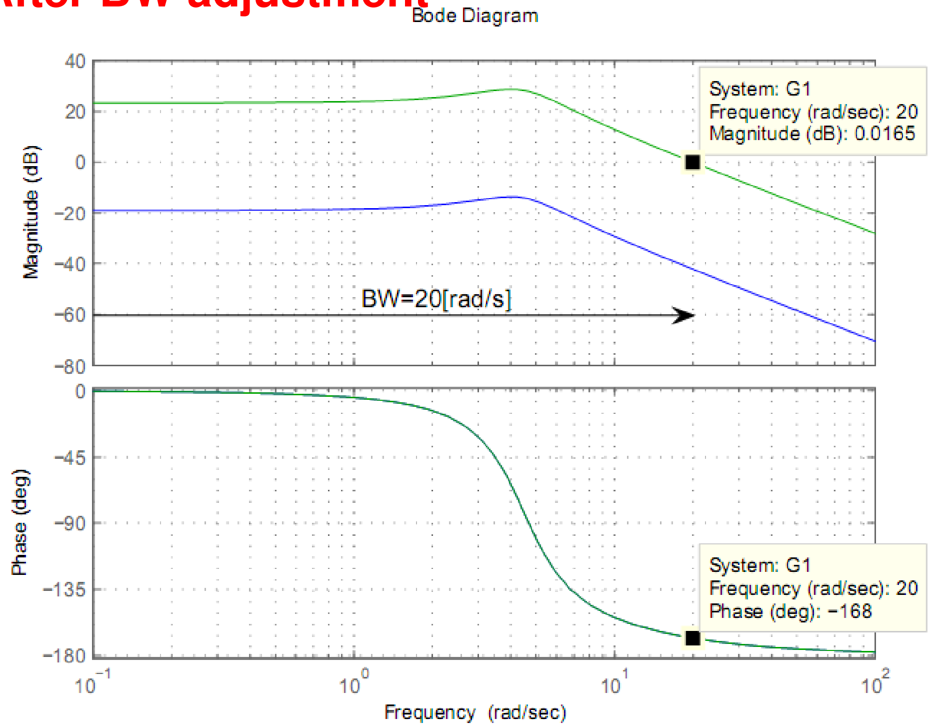


Step 2: Bandwidth Adjustment (Use a forward gain *K*)

$$
\begin{aligned}
K|G(j\omega)|_{\omega=20} &= 1 \\
K\left|\frac{3(j20+3)}{(j20+4)(-20^2+3\times j20+20)}\right| &= 1 \\
K \times 0.0077 &= 1 \\
K &= 129.3
\end{aligned}
$$

```
K=1/gainG
G1=K*G
bode(G1); grid on; hold on;
```

After BW adjustment



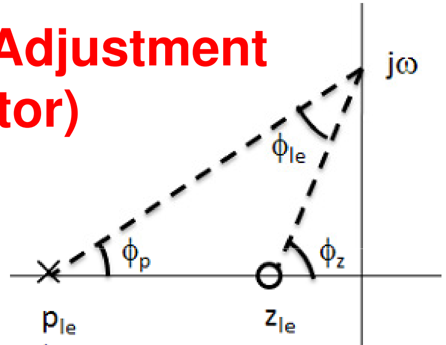
Phase Margin

$$
\begin{aligned}
\angle G(j\omega)_{\omega=20} &= \angle(j20+3) - \angle(j20+4) - \angle(20^2 + 3\times j20 + 20) \\
&= \angle(j20+3) - \angle(j20+4) - \angle(j60 - 380) \\
&= \tan^{-1}\left(\frac{20}{3}\right) - \tan^{-1}\left(\frac{20}{4}\right) - \left[180^\circ - \tan^{-1}\left(\frac{60}{380}\right)\right] \\
&= -168^\circ
\end{aligned}
$$

- Phase margin is $180^o - 168^o = 12^o$ Not adequate ?
- Additional phase required $PM - 12^o = 33^o$
- Use a lead compensator to provide additional phase

Step 3: Phase Margin Adjustment (Use a lead compensator)

$\frac{(s+z_{le})}{(s+p_{le})};\ p_{le} > z_{le}$



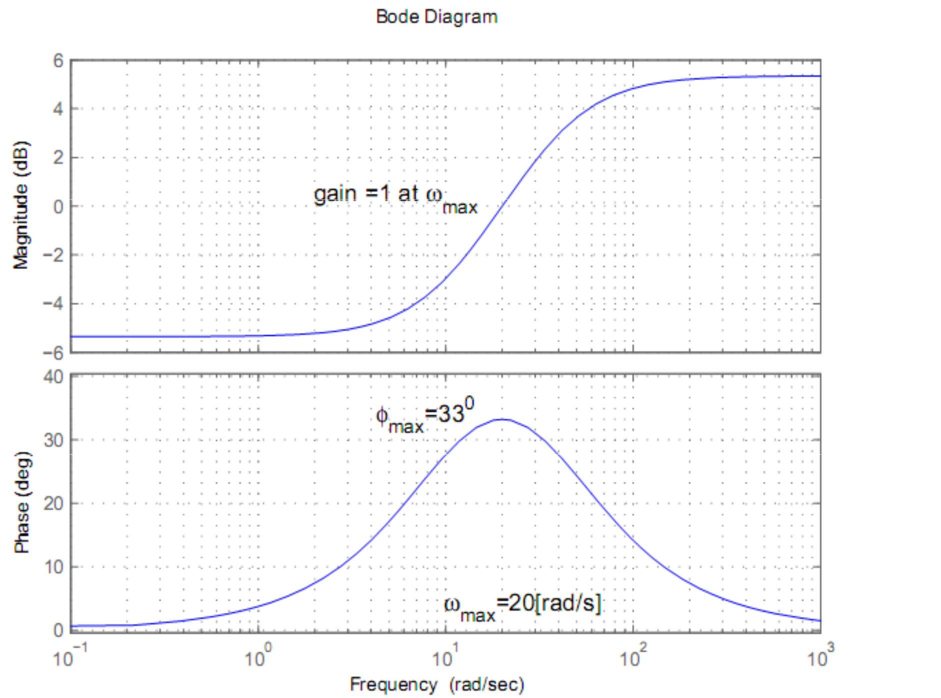
$$\phi_{le}(\omega) = \tan^{-1}\left(\frac{\omega}{z_{le}}\right) - \tan^{-1}\left(\frac{\omega}{p_{le}}\right)$$

$$\omega_{\phi_{max}} = \sqrt{z_{le}p_{le}}$$

$$
\begin{aligned}
\phi_{max} &= \tan^{-1}\left(\frac{\omega_{max}}{z_{le}}\right) - \tan^{-1}\left(\frac{z_{le}}{\omega_{max}}\right) \\
&= \tan^{-1}\left(\frac{\sqrt{z_{le}p_{le}}}{z_{le}}\right) - \tan^{-1}\left(\frac{\sqrt{z_{le}p_{le}}}{p_{le}}\right)
\end{aligned}
$$

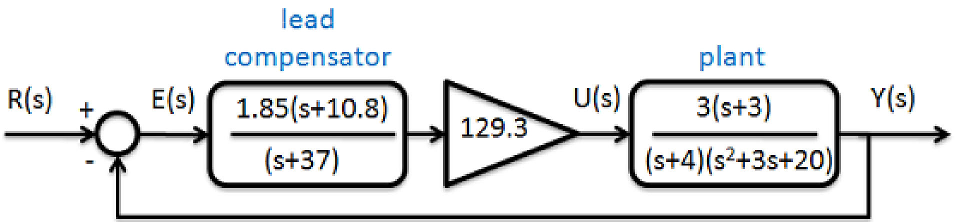
$$33^0 = \tan^{-1}\left(\frac{20}{z_{le}}\right) - \tan^{-1}\left(\frac{z_{le}}{20}\right)$$ ⟵ BW=20

This can be numerically solved and find that $z_{le} \approx 10.8$, and then $p_{le} = (20^2/z_{le}) = 37$

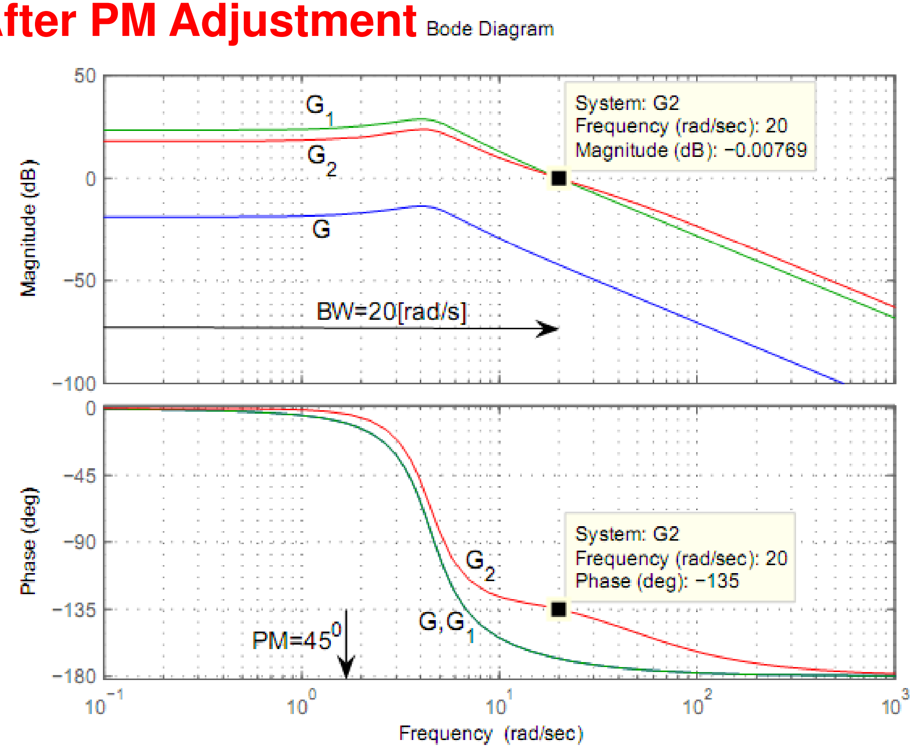


Step 4: Adjust lead compensator gain to unity (Use a gain K_{le})

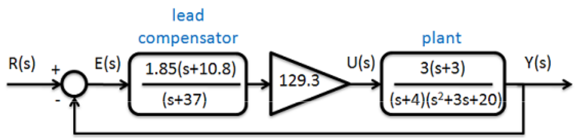
$$
\begin{aligned}
|C_{le}(j20)| &= K_{le}\left|\frac{(s+10.8)}{(s+37)}\right| \\
1 &= K_{le}\frac{\sqrt{20^2+10.8^2}}{\sqrt{20^2+37^2}} \\
1 &= K_{le}0.54 \\
1.85 &= K_{le}
\end{aligned}
$$



After PM Adjustment



Step 5: Steady State Error Correction (Use a lag compensator)



$$
\begin{aligned}
E(s) &= R(s) - Y(s) \\
&= R(s) - E(s)C_{le}(s)KG(s) \\
E(s) &= \frac{1}{1+C_{le}(s)KG(s)}R(s)
\end{aligned}
$$

Steady State Error for Unit Step Input (FVT)

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0}\frac{1}{1+C_{le}(s)KG(s)} \\
&= \lim_{s\to 0}\frac{1}{1+\frac{1.85(s+10.8)}{(s+37)}129.3\frac{(s+3)}{(s^2+3s+20)}} \\
&= \frac{1}{1+10.47} \\
&= 0.087
\end{aligned}
$$

Almost 1%, not acceptable

Step 6: Design a Lag Compensator

$$
C_{la}(s) = \frac{s+z_{la}}{s+p_{la}} \qquad E(s) = \frac{1}{1+C_{la}(s)C_{le}(s)KG(s)}
$$

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} s\frac{1}{1+\frac{(s+z_{la})}{(s+p_{la})}\frac{1.85(s+10.8)}{(s+37)}129.3\frac{3(s+3)}{(s^2+3s+20)}}\frac{1}{s} \\
&= \frac{1}{1+\frac{z_{la}}{p_{la}}10.47} \\
&= \frac{p_{la}}{p_{la}+10.47z_{la}}
\end{aligned}
$$

To meet the requirement

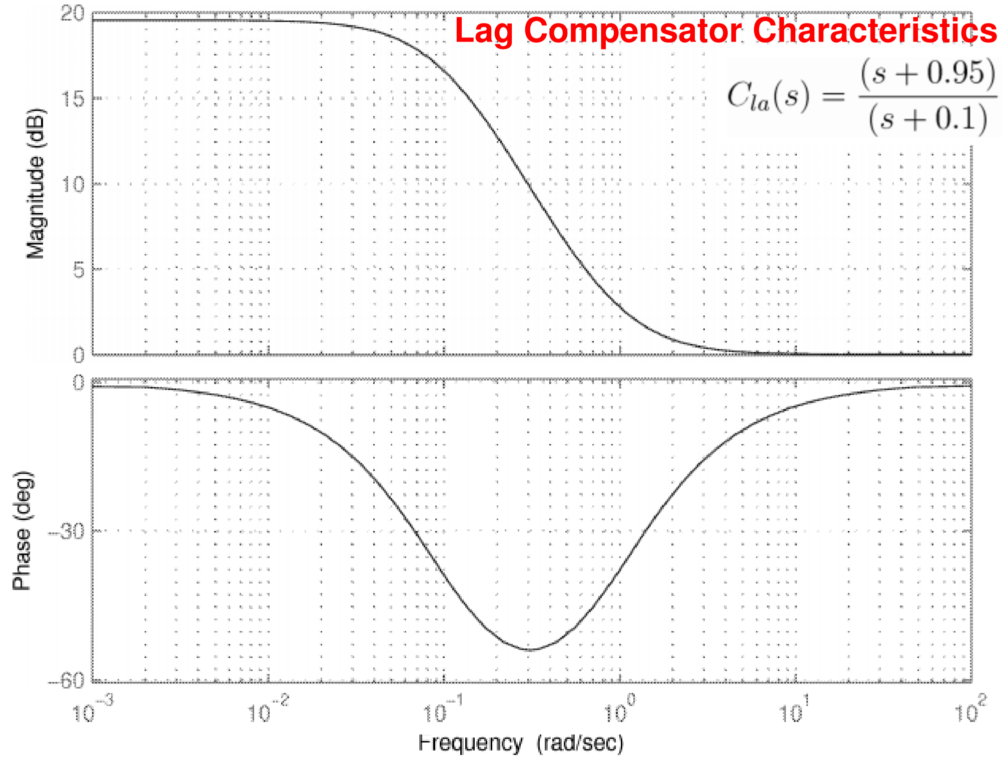
$$
\begin{aligned}
\frac{p_{la}}{p_{la}+10.47z_{la}} &= 0.01 \\
\frac{z_{la}}{p_{la}} &= \frac{0.99}{0.1047} \\
&= 9.46
\end{aligned}
$$

Place the pole and zero near the origin so that the effect is confined to low frequencies

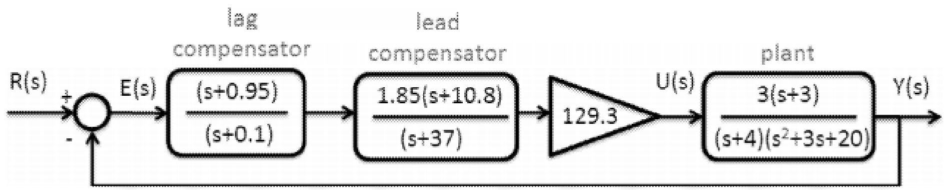
Lets select $p_{la} = 0.1$

then $z_{la} = 0.95$

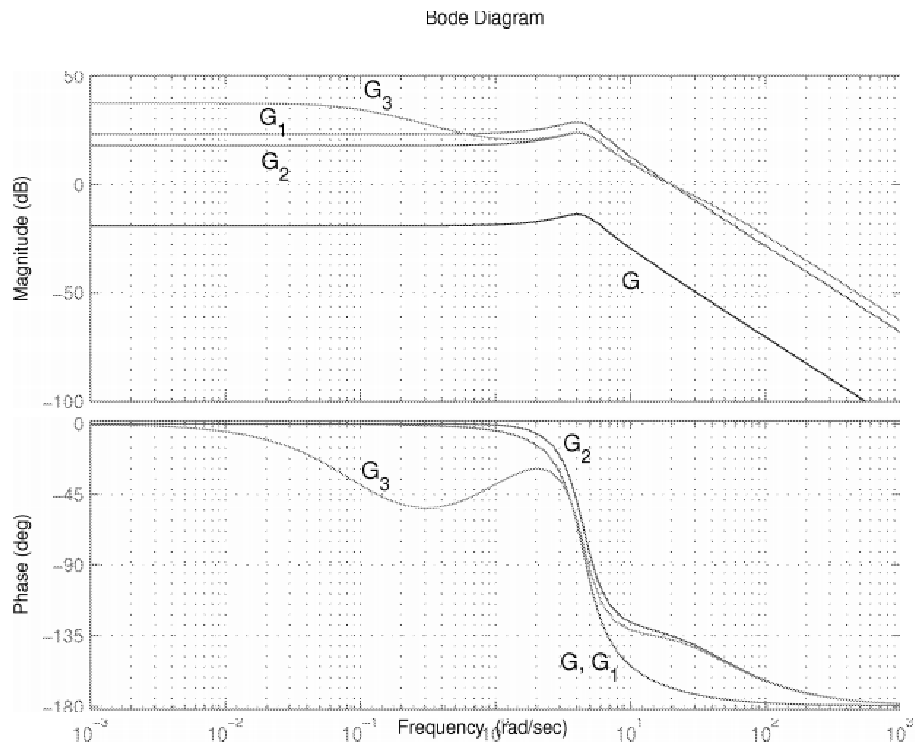
$$C_{la}(s) = \frac{(s+0.95)}{(s+0.1)}$$



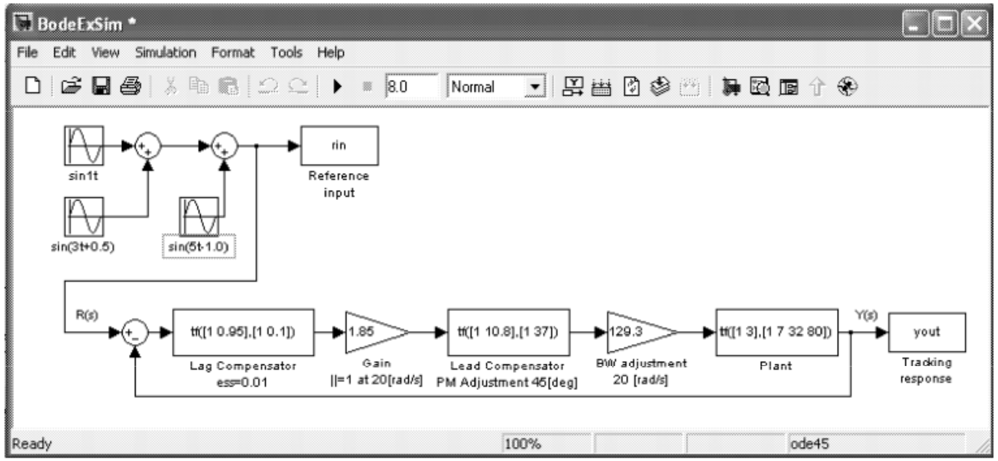
Servo Controller



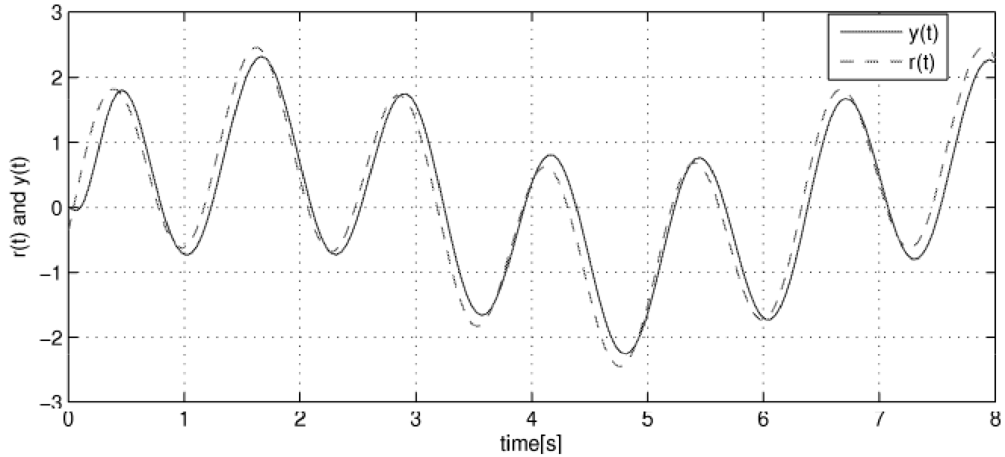
BW=20[rad/s]
PM=45°
Steady state error=0.01



Step 7: Simulation



Simulation Results for 1,3,5 [rad/s]



Simulation Results for 8,10,12 [rad/s]